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Deptt - Mathematics
Topic - Partial Differentiation
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Partial Differentiation

Definition :→ Let $u = f(x,y)$ be a function of two variables x and y

Then $\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x,y)}{h}$

$\frac{\partial u}{\partial x}$ = Partial derivative of u with respect to x only

$\frac{\partial u}{\partial x}$ means that u has been differentiated with respect to x only and remaining variable has been treated as constant.

$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{f(x, y+k) - f(x,y)}{k}$ if this limit exist.

$\frac{\partial u}{\partial y}$ = partial derivative of u with respect to y only.

Ex:→ If $u = x^3 + y^3 + 3axy$

then $\frac{\partial u}{\partial x} = 3x^2 + 3ay$ and $\frac{\partial u}{\partial y} = 3y^2 + 3ax$

Partial Derivative of Higher order

Partial Derivative of 2nd order

$\frac{\partial^2 u}{\partial x^2}$ ⇒ $\frac{\partial u}{\partial x}$ is differentiated with respect x only and other variables is treated as constant.

$\frac{\partial^2 u}{\partial y^2}$ ⇒ $\frac{\partial u}{\partial y}$ is differentiated with respect to y only and other variable has been treated as constant.

$\frac{\partial^2 u}{\partial x \partial y}$ ⇒ $\frac{\partial u}{\partial x}$ is differentiated with respect to y and in this case x has been treated as constant

$\frac{\partial^2 u}{\partial y \partial x}$ ⇒ $\frac{\partial u}{\partial y}$ is differentiated with respect to x and in this case y has been treated as constant.

Ex :→ If $u = \log \frac{x^2+y^2}{xy}$ verify $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

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By the question

$u = \log \frac{x^2+y^2}{xy}$ then we have to verify $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

$\Rightarrow u = \log(x^2+y^2) - \log x - \log y$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x - \frac{1}{x} - 0$$

$$= \frac{2x}{x^2+y^2} - x^{-1}$$

$$= 2x(x^2+y^2)^{-1} - x^{-1}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} = 2x \frac{\partial}{\partial y}(x^2+y^2)^{-1} - \frac{\partial}{\partial x} x^{-1}$$

$$= -\frac{2x}{(x^2+y^2)^2} \cdot 2y - 0$$

$$= -\frac{4xy}{(x^2+y^2)^2} \text{ ——— (I)}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2} \cdot 2y - \frac{1}{y}$$

$$= 2y \cdot (x^2+y^2)^{-1} - y^{-1}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y} = 2y \cdot \left[-\frac{1}{(x^2+y^2)^2} \cdot 2x - 0\right]$$

$$= -\frac{4xy}{(x^2+y^2)^2} \text{ ——— (II)}$$

From (I) and (II)

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = -\frac{4xy}{(x^2+y^2)^2} \quad \text{(verified)}$$

Problem :→ If $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$ prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}$

Here $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \cdot \tan^{-1}\frac{x}{y}$

$$\frac{\partial u}{\partial y} = x^2\left\{\frac{1}{1+y^2/x^2} \cdot \frac{1}{x}\right\} - \left\{2y \cdot \tan^{-1}\frac{x}{y} + y^2 \cdot \frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right)\right\}$$

$$= x^2 \cdot \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} - 2y \tan^{-1}\frac{x}{y} + \frac{xy^2}{x^2+y^2}$$

$$= \frac{x^4}{x(x^2+y^2)} - 2y \tan^{-1}\frac{x}{y} + \frac{xy^2}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y \cdot \tan^{-1}\frac{x}{y}$$

$$= \frac{x(x^2+y^2)}{x^2+y^2} - 2y \cdot \tan^{-1}\frac{x}{y}$$

$$= x - 2y \cdot \tan^{-1}\frac{x}{y}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 1 - 2y \cdot \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y}$$

$$= 1 - \frac{2y^2}{x^2+y^2}$$

$$= \frac{x^2+y^2-2y^2}{x^2+y^2}$$

$$= \frac{x^2-y^2}{x^2+y^2}$$

If $u = e^{xyz}$, show that-

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz}(1 + 3xyz + x^2y^2z^2)$$

$$u = e^{xyz}$$

$$\frac{\partial u}{\partial z} = \frac{\partial e^{xyz}}{\partial xyz} \cdot xy \cdot \frac{\partial z}{\partial z}$$

$$= e^{xyz} \cdot xy$$

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial y}\left[xy\, e^{xyz}\right]$$

$$= x \frac{\partial}{\partial y}\left[y \cdot e^{xyz}\right]$$

$$= x\left[e^{xyz} + y \cdot e^{xyz} \cdot xz\right]$$

$$= \cancel{xe^{xyz}}\; x(1+xyz) \cdot e^{xyz}.$$

$$\frac{\partial^2 u}{\partial y \partial z} = (x + x^2yz)e^{xyz}$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}[x + x^2yz]e^{xyz}$$

$$= [1 + 2xyz]e^{xyz} + (x + x^2yz) \cdot e^{xyz} \cdot yz]$$

$$= [1 + 2yzx + xyz + x^2y^2z^2]e^{xyz}$$

$$= [1 + 3xyz + x^2y^2z^2]e^{xyz}$$

Problem If $u = f(y/x)$ show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Here $u = f(y/x)$

$$\frac{\partial u}{\partial x} = -f'(y/x)\frac{y}{x^2}$$

$$= -\frac{y}{x^2}f'(y/x)$$

$$x\frac{\partial u}{\partial x} = -y/x\, f'(y/x)$$

$$\frac{\partial u}{\partial y} = f'(y/x) \cdot \frac{1}{x}$$

$$y\frac{\partial u}{\partial y} = \frac{y}{x}f'(y/x)$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{y}{x}f'(y/x) + y/x\, f'(y/x)$$

$$= 0$$

If $u = \sin^{-1}\frac{x}{y} + \tan^{-1} y/x$ Prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

$$u = \sin^{-1}\frac{x}{y} + \tan^{-1} y/x$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-x^2/y^2}}\cdot\frac{1}{y} + \frac{1}{1+y^2/x^2}\cdot\left(-\frac{y}{x^2}\right)$$

$$= \frac{y}{\sqrt{y^2-x^2}}\cdot\frac{1}{y} - \frac{\cancel{x^2}}{x^2+y^2}\,\frac{y}{\cancel{x^2}}$$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2}$$

Again

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-x^2/y^2}}\cdot\left(\frac{-x}{y^2}\right) + \frac{1}{1+y^2/x^2}\cdot\frac{1}{x}$$

$$= -\frac{y}{\sqrt{y^2-x^2}}\cdot\frac{x}{y^2} + \frac{x}{x^2+y^2}$$

$$\therefore y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \cancel{\frac{x}{\sqrt{y^2-x^2}}} - \cancel{\frac{xy}{x^2+y^2}} - \cancel{\frac{x}{\sqrt{y^2-x^2}}} + \cancel{\frac{xy}{x^2+y^2}}$$

$$= 0$$

If $u = \log \frac{x^2+y^2}{x+y}$, Prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$$

Sol^n :→ Here $u = \log \frac{x^2+y^2}{x+y}$

$$= \log(x^2+y^2) - \log(x+y)$$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x - \frac{1}{x+y} \cdot 1$$

$$x\frac{\partial u}{\partial x} = \frac{2x^2}{x^2+y^2} - \frac{x}{x+y}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2} \cdot 2y - \frac{1}{x+y} \cdot 1$$

$$y\frac{\partial u}{\partial y} = \frac{2y^2}{x^2+y^2} - \frac{y}{x+y}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2x^2}{x^2+y^2} - \frac{x}{x+y} + \frac{2y^2}{x^2+y^2} - \frac{y}{x+y}$$

$$= 2\left[\frac{x^2+y^2}{x^2+y^2}\right] - \left[\frac{x+y}{x+y}\right]$$

$$= 2-1 = 1$$

Problem If $u = (x^2+y^2+z^2)^{-1/2}$ Show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u$$

Sol^n we have

$$u = (x^2+y^2+z^2)^{-1/2}$$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x$$

$$\frac{\partial u}{\partial x} = -\frac{x}{(x^2+y^2+z^2)^{3/2}}$$

$$x\frac{\partial u}{\partial x} = -\frac{x^2}{(x^2+y^2+z^2)^{3/2}}$$

Similarly

$$y\frac{\partial u}{\partial y} = -\frac{y^2}{(x^2+y^2+z^2)^{3/2}}$$

and $$z\frac{\partial u}{\partial z} = -\frac{z^2}{(x^2+y^2+z^2)^{3/2}}$$

Now $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = -\left[\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= -\frac{1}{(x^2+y^2+z^2)^{1/2}}$$

$$= -(x^2+y^2+z^2)^{-1/2}$$

$$= -u$$

thus $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = -u$$

Problem :→ If

$u = 2(ax+by)^2 - (x^2+y^2)$ and $a^2+b^2=1$

prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Solution :→ Here

$$u = 2(ax+by)^2 - (x^2+y^2)$$

$$\frac{\partial u}{\partial x} = 4(ax+by)\cdot a - 2x$$

$$\frac{\partial^2 u}{\partial x^2} = 4a^2-2$$

Also $$\frac{\partial u}{\partial y} = 4(ax+by)\cdot b - 2y$$

$$\frac{\partial^2 u}{\partial y^2} = 4b\cdot b - 2 = 4b^2-2$$

Now $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 4a^2-2+4b^2-2$$

$$= 4(a^2+b^2)-4$$

$$= 4\times 1 - 4 = 0$$

Problem :→ If $u = \tan^{-1} y/x$

prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

Here $u = \tan^{-1} y/x$

$$\frac{\partial u}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)$$

$$= \frac{1}{\frac{x^2+y^2}{x^2}}\left(-\frac{y}{x^2}\right)$$

$$= \frac{-y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = -y\,(-1)\,(x^2+y^2)^{-2}\cdot 2x$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+y^2/x^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = -x\,(x^2+y^2)^{-2}\cdot 2y$$

$$= -\frac{2xy}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2}$$

$$= 0$$

Problem :→ If $u = \log(x^3+y^3+z^3-3xyz)$, show that

(i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$

(ii) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$

Solution :→
Here $u = \log(x^3+y^3+z^3-3xyz)$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}\cdot(3x^2-3yz)$$

Thus $\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3+y^3+z^3-3xyz}$

Similary $\frac{\partial u}{\partial y} = \frac{3y^2 - 3zx}{x^3+y^3+z^3-3xyz}$

$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3+y^3+z^3-3xyz}$

$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2) - 3(yz+zx+xy)}{x^3+y^3+z^3-3xyz}$

$= \frac{3[x^2+y^2+z^2 - xy - yz - zx]}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$

$[a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)]$

$= \frac{3}{x+y+z}$

Now (ii) we are to show that-

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$

Now $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u$

$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$

$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \cdot \frac{3}{x+y+z}$

$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$

$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$

$= -\frac{9}{(x+y+z)^2}$

Chas $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

Proved

Problem :→

If $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$ Prove that

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right]$$

Solution :→ By the question.

$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$$

Diff. with respect to x partially

$$\left[\frac{2x}{a^2+u}-\frac{x^2}{(a^2+u)^2}\frac{\partial u}{\partial x}\right]+\left[\frac{-y^2}{(b^2+u)^2}\frac{\partial u}{\partial x}\right]+\left[\frac{-z^2}{(c^2+u)^2}\cdot\frac{\partial u}{\partial x}\right]=0$$

$$\frac{2x}{a^2+u}-\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial x}=0 \quad (1)$$

$$\Rightarrow \left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial x}=\frac{2x}{a^2+u}$$

$$\therefore \frac{\partial u}{\partial x}=\frac{2x}{a^2+u}\Big/\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)^2=\frac{4x^2}{(a^2+u)^2}\Big/\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]^2 \quad (2)$$

Similarly

$$\left(\frac{\partial u}{\partial y}\right)^2=\frac{4y^2}{(b^2+u)^2}\Big/\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]^2$$

$\therefore \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$

proved

Problem :→

If $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$ Prove that–

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right]$$

Solution :→ By the question.

$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$$

Diff. with respect to x partially

$$\left[\frac{2x}{a^2+u}-\frac{x^2}{(a^2+u)^2}\frac{\partial u}{\partial x}\right]+\left[\frac{-y^2}{(b^2+u)^2}\frac{\partial u}{\partial x}\right]+\left[\frac{-z^2}{(c^2+u)^2}\cdot\frac{\partial u}{\partial x}\right]=0$$

$$\frac{2x}{a^2+u}-\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial x}=0 \quad (1)$$

$$\Rightarrow \left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial x}=\frac{2x}{a^2+u}$$

$$\therefore \frac{\partial u}{\partial x}=\frac{2x}{a^2+u}\Bigg/\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)^2=\frac{4x^2}{(a^2+u)^2}\Bigg/\left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\}^2$$

Similarly

$$\left(\frac{\partial u}{\partial y}\right)^2=\frac{4y^2}{(b^2+u)^2}\Bigg/\left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\}^2 \quad (2)$$

and

$$\left(\frac{\partial u}{\partial z}\right)^2 = \frac{4z^2}{(c^2+u)^2} \Big/ \left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right\}^2$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2$$

$$= 4\left(\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right) \Big/ \left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right\}^2$$

$$= \frac{4}{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}}$$

Also $x\frac{\partial u}{\partial x} = \frac{2x^2}{a^2+u} \Big/ \frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}$

$y\frac{\partial u}{\partial y} = \frac{2y^2}{b^2+u} \Big/ \frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}$

and $z\frac{\partial u}{\partial z} = \frac{2z^2}{c^2+u} \Big/ \frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}$

$\therefore$ Now $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2\,\Sigma\frac{x^2}{a^2+u} \Big/ \Sigma\frac{x^2}{(a^2+u)^2}$

$$= \frac{2}{\Sigma\frac{x^2}{(a^2+u)^2}} \quad \text{——} \; (3)$$

But $\Sigma\frac{x^2}{a^2+u} = 1$

$\therefore$ From (2) and (3)

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)$$

proved